Linear Algebra

[KOMS120301] - 2023/2024

7.1 - Vectors in R^n

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Learning objectives

After this lecture, you should be able to:

1. explain the definition of vectors in general;
2. explain the definition of vectors in Linear Algebra;
3. explain some operations on vectors, such as:
   - vector addition and scalar multiplication;
   - linear combination;

Part 1: **Vectors** (*in general*)

What is a vector?

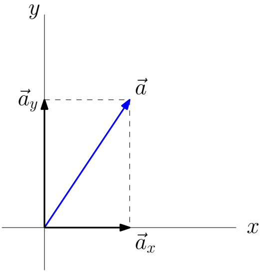
Three ways of defining vectors:

1. Physics perspective
2. Mathematics perspective
3. CS perspective

What is a vector (in physics)?

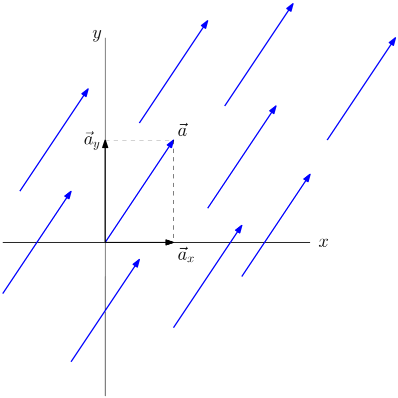
Vectors are arrows pointing in space. They are quantities that possess both *magnitude* and *direction*; e.g. force, velocity.

Usually, denoted by a letter typed in bold, or with an arrow above it; e.g. $\vec{a}$. It is often drawn as an arrow having appropriate length and direction .



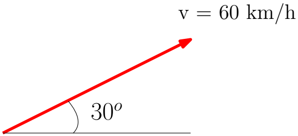
What defined a vector (in physics)?

- Length (magnitude)
- Direction



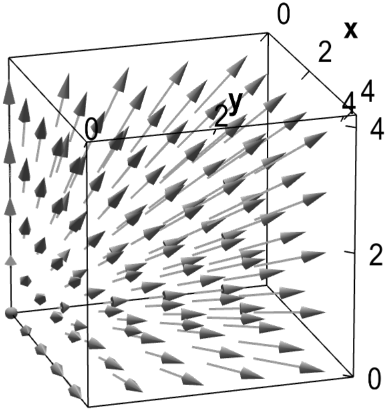
Two vectors are the same if they have the same length and direction

Example of vector in Physics



The velocity of a car is $60km/h$, and it goes to 30^o in the north-east direction.

Vectors in 3D-space (in physics)



What is a vector (in CS)?

Example

A teacher needs to check their students health, by measuring their *weight* and *height*. How should the data be represented?

$$\begin{bmatrix} 40\ kg \\ 150\ cm \end{bmatrix}$$ This is a 2D vector

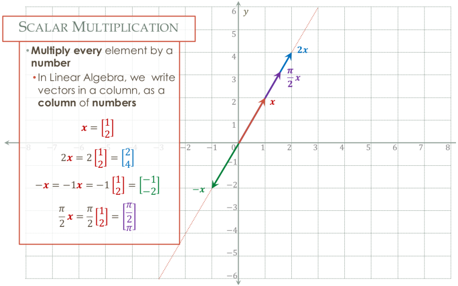
$$\begin{bmatrix} 40\ kg \\ 150\ cm \\ 14\ years \end{bmatrix}$$ This is a 3D vector

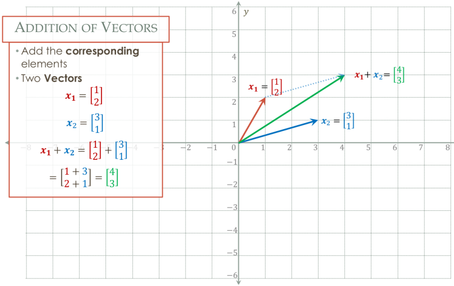
In CS, a vector can be considered as a list (tuples) of numbers

What is a vector (in Mathematics)?

The mathematical concept of vectors are combination of the two:

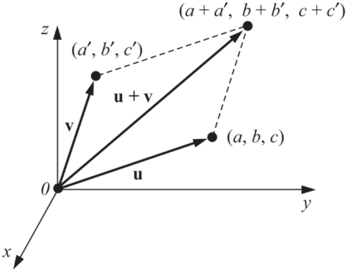
- Vectors can be viewed geometrically or algebraically;
- We can perform operations such as addition, multiplication, substraction, etc.



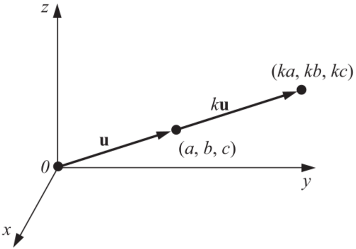


Back to high school: *simple operations in vectors you might have learned in physics*

1. Vectors addition
2. Scalar multiplication

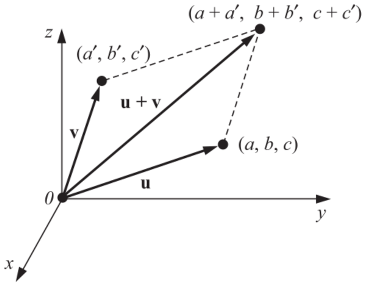


(a) Vector Addition



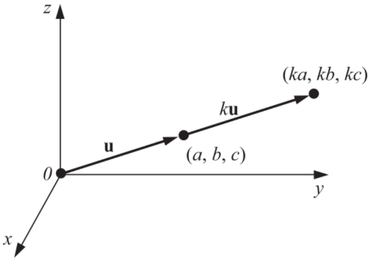
(b) Scalar Multiplication

Vectors addition ($\mathbf{u} + \mathbf{v}$)



- Geometrically, the *resultant* $\mathbf{u} + \mathbf{v}$ is obtained by the parallelogram law
- If $\mathbf{u}$ has endpoints (a, b, c) and $\mathbf{v}$ has endpoints (a', b', c'), then $\mathbf{u} + \mathbf{v}$ has endpoints $(a + a', b + b', c + c')$

Scalar multiplication ($k\mathbf{u}$)

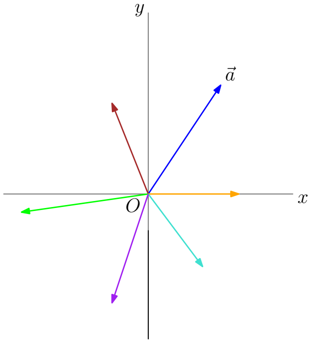


- Let $k \in \mathbb{R}$, then $k\mathbf{u}$ is the vector having magnitude k times the magnitude of u, and same direction when $k > 0$ or the opposite direction when $k < 0$.
- If $\mathbf{u}$ has endpoints (a, b, c), then the endpoints of $k\mathbf{u}$ are (ka, kb, kc).

Part 2: **Vectors in Linear Algebra**

Vectors in Linear Algebra

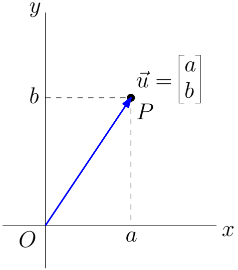
Geometrically:



- Vectors are arrows originated at the origin O
- Notations: **u,v,w, . . .** or $\vec{u}, \vec{v}, \vec{w}, \ldots$

Vectors in Linear Algebra

In 2D



Vectors are arrows originated at the origin O.

It is not the same as a point.

Vector $\vec{u}$ is equivalent to $\overrightarrow{OP}$

The number a and b in $\begin{bmatrix} a \\ b \end{bmatrix}$ indicate how far the vector $\vec{u}$ moves along the x-axis and the y-axis resp.

The positive (resp. negative) sign of a or b indicates that it moves toward the right or up (resp. left or down).

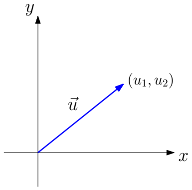
In 3D, it is similar, but we consider three axes (x, y, and z).

What is a vector space?

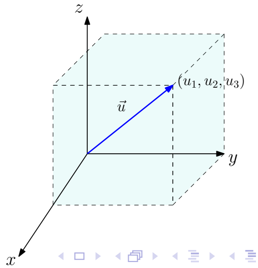
- An ordered n-tuple is a sequence of *real numbers*: $(a_1, a_2, \dots, a_n)$ (or, can be seen as a vector).
- An n-space is a set of all n-tuples of real numbers. Usually denoted as $\mathbb{R}^n$. For $n = 1$, $\mathbb{R}^1 \equiv \mathbb{R}$.
  - This space is where vectors are defined
- The space is also called Euclidean space.

Example:

Vector in $\mathbb{R}^2$

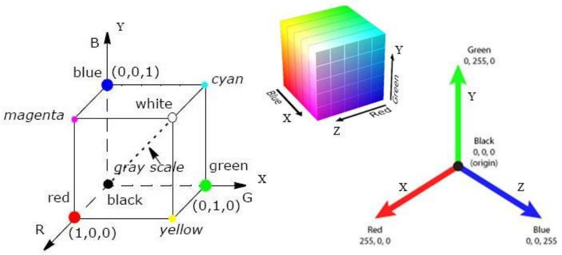


Vector in $\mathbb{R}^3$



Example

1. $\vec{u} = (3, 6) \;\rightarrow\;$ vector in $\mathbb{R}^2$
2. $\vec{v} = (2, -4, 5) \;\rightarrow\;$ vector in $\mathbb{R}^4$
3. $\vec{w} = (-4, 2, -3, 1) \;\rightarrow\;$ vector in $\mathbb{R}^4$
4. $\vec{c} = (r, g, b) \;\rightarrow\;$ vector in RGB-model



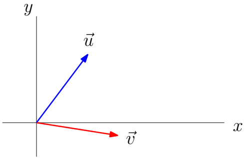
We will go back to the vector space $\mathbb{R}^n$.

For now, let us look at $\mathbb{R}^2$ and $\mathbb{R}^3$.

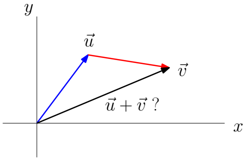
Part : Vector operations in R_2 and R_3

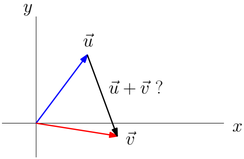
Vectors addition (geometric representation)

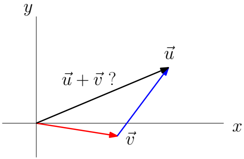
Let us given the following vectors:



Which one defines $\vec{u} + \vec{v}$?



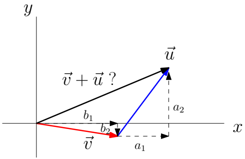


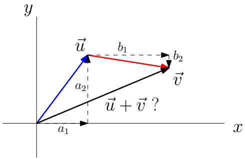


Vectors addition (geometric representation)

A vector defines a certain movement in space (how far, which direction).

- $\vec{u} = [a_1 \ \ a_2] \rightarrow$ moving a_1 steps in the x-axis direction, and a_2 steps in the y-axis direction.
- $\vec{v} = [b_1 \ \ b_2] \rightarrow$ moving b_1 steps in the x-axis direction, and b_2 steps in the y-axis direction.

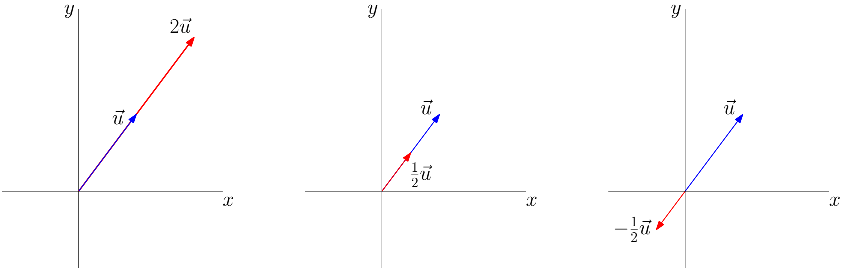




So $\vec{u} + \vec{v}$ can be seen as moving along vector $\vec{u}$ continued by moving along vector $\vec{v}$, i.e. moving $a_1 + b_1$ steps in the x-axis direction, and $a_2 + b_2$ steps in the y-axis direction.

$$\vec{u} + \vec{v} = [(a_1 + b_1) \ \ (a_2 + b_2)]$$

Scalar multiplication (geometric representation)



Multiplying a vector by a scalar can be seen as “scaling” a vector (stretching, and sometimes reversing the direction of a vector).

Exercise

Give two vectors at $\mathbb{R}^2$.

- Calculate the sum of the two vectors.
- Geometrically draw the two vectors and their resultant on the Cartesian plane.
- Multiply one vector by a scalar $\mathbb{R}^+$ and the other by a scalar $\mathbb{R}^-$.
- Draw both result vectors on the $\mathbb{R}^2$ field.

Part : Spatial Vectors

Vectors in $\mathbb{R}^3$

Vectors in $\mathbb{R}^3$ are called spatial vectors, appear in many applications, especially in physics.

Special notation:

- $\mathbf{i} = [1, 0, 0]$ denotes the unit vector in the x-direction
- $\mathbf{j} = [0, 1, 0]$ denotes the unit vector in the y-direction
- $\mathbf{k} = [0, 0, 1]$ denotes the unit vector in the z-direction

Any vector $\mathbf{u} = [a, b, c]$ in $\mathbb{R}^3$ can be expressed uniquely in the form:

$$\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Vectors in $\mathbb{R}^3$

Important! $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are vectors, and they are unit vectors. Furthermore:

$$\mathbf{i} \cdot \mathbf{i} = 1,\ \mathbf{j} \cdot \mathbf{j} = 1,\ \mathbf{k} \cdot \mathbf{k} = 1 \quad \textit{and} \quad \mathbf{i} \cdot \mathbf{j} = 0,\ \mathbf{i} \cdot \mathbf{k} = 0,\ \mathbf{j} \cdot \mathbf{k} = 0$$

The right equality shows that $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are orthogonal one to each other.

All vector operations still hold:

For $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then:

- $\mathbf{u} + \mathbf{v} = (u_1 + v_1)\mathbf{i} + (u_2 + v_2)\mathbf{j} + (u_3 + v_3)\mathbf{k}$
- $k\mathbf{u} = ku_1\mathbf{i} + ku_2\mathbf{j} + ku_3\mathbf{k}$ for any $k \in \mathbb{R}$
- $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$
- $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$

Example

Let $\mathbf{u} = 3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} - 8\mathbf{j} + 5\mathbf{k}$. Find $3\mathbf{u} - 2\mathbf{v}$.

$$\begin{aligned} 3\mathbf{u} - 2\mathbf{v} &= 3(3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}) - 2(4\mathbf{i} - 8\mathbf{j} + 5\mathbf{k}) \\ &= (9\mathbf{i} + 15\mathbf{j} - 6\mathbf{k}) + (-8\mathbf{i} + 16\mathbf{j} - 10\mathbf{k}) \\ &= 1\mathbf{i} + 31\mathbf{j} - 16\mathbf{k} \end{aligned}$$

to be continued...